Burgers velocity fields and dynamical transport processes

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Abstract

We explore a connection of the forced Burgers equation with the Schrödinger (diffusive) interpolating dynamics in the presence of deterministic external forces. This entails an exploration of the consistency conditions that allow to interpret dispersion of passive contaminants in the Burgers flow as a Markovian diffusion process. In general, the usage of a continuity equation $\partial_t \rho = -\nabla(\vec{v}\rho)$, where $\vec{v} = \vec{v}(\vec{x}, t)$ stands for the Burgers field and ρ is the density of transported matter, is at variance with the explicit diffusion scenario. Under these circumstances, we give a complete characterisation of the diffusive matter transport that is governed by Burgers velocity fields. The result extends both to the approximate description of the transport driven by an incompressible fluid and to motions in an infinitely compressible medium.

The Burgers equation with (typically without, [1, 2]) the forcing term $\vec{F}(\vec{x},t)$:

$$\partial_t \vec{v} + (\vec{v}\nabla)\vec{v} = \nu \triangle \vec{v} + \vec{F}(\vec{x},t) \tag{1}$$

and especially its statistically relevant $curl\, \vec{v} = 0$ solutions (in below we shall not use random initial data), recently have acquired a considerable popularity in various physical contexts, [3]-[19].

Although Burgers velocity fields can be analysed on their own, frequently one needs a supplementary insight into the matter transport dynamics that is consistent with the chosen (Burgers) velocity field evolution. Then, the passive scalar (tracer or contaminant) advection-in-a-flow problem, [14, 11, 16] naturally appears through the parabolic dynamics:

$$\partial_t T + (\vec{v}\nabla)T = \nu \triangle T \tag{2}$$

While looking for the stochastic implementation of the microscopic (molecular) dynamics (2), [21, 11, 16], it is assumed that the "diffusing scalar" (contaminant in the lore of early statistical turbulence models) obeys an Itô equation:

$$d\vec{X}(t) = \vec{v}(\vec{x},t)dt + \sqrt{2\nu}d\vec{W}(t) \tag{3}$$

$$\vec{X}(0) = \vec{x}_0 \to \vec{X}(t) = \vec{x}$$

where the given forced Burgers velocity field is perturbed by the noise term representing a molecular diffusion. In the Itô representation of diffusion-type random variable $\vec{X}(t)$ one explicitly refers to the Wiener process $\sqrt{2\nu}\vec{W}(t)$, instead of the usually adopted formal white noise integral $\int_0^t \vec{\eta}(s)ds$, coming from the Langevin-type version of (3).

Under these premises, we cannot view equations (1)-(3) as completely independent (disjoint) problems: the velocity field $\vec{v}$ cannot be arbitrarily inferred from (1) or any other velocity-defining equation without verifying the *consistency* conditions, which would allow to associate (2) and (3) with a well defined random dynamics (stochastic process), and Markovian diffusion in particular, [22, 23].

In connection with the usage of Burgers velocity fields (with or without external forcing) which in (3) clearly are intended to replace the standard *forward drift* of the would-be-involved Markov diffusion process, we have not found in the literature any attempt to resolve apparent contradictions arising if (2) and/or (3) are defined by means of (1). Also, an issue of the necessary *correlation* (cf. [16], Chap.7.3, devoted to the turbulent transport and the related dispersion of contaminants) between the probabilistic Fokker-Planck dynamics of the diffusing tracer, and this of the passive tracer (contaminant) concentration (2), has been left aside in the literature.

Moreover, rather obvious hesitation could have been observed in attempts to establish the most appropriate matter transport rule, if (1)-(3) are adopted. Depending on the particular phenomenological departure point, one either adopts the standard continuity equation, [3, 4], that is certainly valid to a high degree of accuracy in the low viscosity limit $\nu \downarrow 0$ of (1)-(3), but incorrect on mathematical grounds *if* there is a diffusion involved *and* simultaneously a solution of (1) stands for the respective *current* velocity of the flow: $\partial_t \rho(\vec{x}, t) = -\nabla[\vec{v}(\vec{x}, t)\rho(\vec{x}, t)]$. Alternatively, following the white noise calculus tradition telling that the stochastic integral $\vec{X}(t) = \int_0^t \vec{v}(\vec{X}(s), s)ds + \int_0^t \vec{\eta}(s)ds$ necessarily implies the Fokker-Planck equation, one adopts, [21]: $\partial_t \rho(\vec{x}, t) = \nu \triangle \rho(\vec{x}, t) - \nabla[\vec{v}(\vec{x}, t)\rho(\vec{x}, t)]$ which is clearly problematic in view of the classic Mc Kean's discussion of the propagation of chaos for the Burgers equation, [24, 25, 26] and the derivation of the stochastic "Burgers process" in this context: "the fun begins in trying to describe this Burgers motion as the path of a tagged molecule in an infinite bath of like molecules", [24].

To put things on the solid ground, let us consider a Markovian diffusion process, which is characterised by the transition probability density (generally inhomogeneous in space and time law of random displacements) $p(\vec{y}, s, \vec{x}, t)$, $0 \leq s < t \leq T$, and the probability density $\rho(\vec{x}, t)$ of its random variable $\vec{X}(t)$, $0 \leq t \leq T$. The process is completely determined by these data. For clarity of discussion, we do not impose any spatial boundary restrictions, nor fix any concrete limiting value of T which, in principle, can be moved to infinity.

The conditions valid for any $\epsilon > 0$:
(a) there holds $lim_{t\downarrow s} \frac{1}{t-s} \int_{|\vec{y}-\vec{x}|>\epsilon} p(\vec{y}, s, \vec{x}, t) d^3x = 0$,
(b) there exists a (forward) drift $\vec{b}(\vec{x}, s) = lim_{t\downarrow s} \frac{1}{t-s} \int_{|\vec{y}-\vec{x}|\leq\epsilon} (\vec{y} - \vec{x}) p(\vec{x}, s, \vec{y}, t) d^3y$,
(c) there exists a diffusion function (in our case it is simply a diffusion coefficient ν) $a(\vec{x}, s) = lim_{t\downarrow s} \frac{1}{t-s} \int_{|\vec{y}-\vec{x}|\leq\epsilon} (\vec{y} - \vec{x})^2 p(\vec{x}, s, \vec{y}, t) d^3y$,
are conventionally interpreted to define a diffusion process, [23, 22]. Under suitable restrictions (boundedness of involved functions, their continuous differentiability) the function:

$$g(\vec{x}, s) = \int p(\vec{x}, s, \vec{y}, T) g(\vec{y}, T) d^3y \tag{4}$$

satisfies the backward diffusion equation (notice that the minus sign appears, in comparison with (2))

$$-\partial_s g(\vec{x}, s) = \nu \triangle g(\vec{x}, s) + [\vec{b}(\vec{x}, s)\nabla] g(\vec{x}, s) \ . \tag{5}$$

Let us point out that the validity of (5) is known to be a *necessary* condition for the existence of a Markov diffusion process, whose probability density $\rho(\vec{x}, t)$ is to obey the Fokker-Planck equation (the forward drift $\vec{b}(\vec{x}, t)$ replaces the previously utilized

Burgers field $\vec{v}(\vec{x},t))$):

$$\partial_t \rho(\vec{x},t) = \nu \triangle \rho(\vec{x},t) - \nabla[\vec{b}(\vec{x},t)\rho(\vec{x},t)] \tag{6}$$

The case of particular interest in the nonequilibrium statistical physics literature appears when $p(\vec{y},s,\vec{x},t)$ is a *fundamental solution* of (5) with respect to variables $\vec{y},s$, [27, 22, 23], see however [28] for an alternative situation. Then, the transition probability density satisfies *also* the second Kolmogorov (e.g. the Fokker-Planck) equation in the remaining $\vec{x},t$ pair of variables. Let us emphasize that these two equations form an *adjoint pair* , referring to the slightly counterintuitive for physicists, although transparent for mathematicians, [30, 31, 32, 33, 34], issue of time reversal of diffusions.

After adjusting (3) to the present context, $\vec{X}(t) = \int_0^t \vec{b}(\vec{X}(s),s)\,ds + \sqrt{2\nu}\vec{W}(t)$ we can utilize standard rules of the Itô stochastic calculus, [35, 32, 33, 34], to realise that for any smooth function $f(\vec{x},t)$ of the random variable $\vec{X}(t)$ the conditional expectation value:

$$lim_{\triangle t \downarrow 0} \frac{1}{\triangle t}[\int p(\vec{x},t,\vec{y},t+\triangle t) f(\vec{y},t+\triangle t) d^3y - f(\vec{x},t)] = (D_+ f)(\vec{X}(t),t) = \tag{7}$$

$$= (\partial_t + (\vec{b}\nabla) + \nu \triangle) f(\vec{x},t)$$

$$\vec{X}(t) = \vec{x}$$

determines the forward drift $\vec{b}(\vec{x},t)$ (if we set components of $\vec{X}$ instead of f) and allows to introduce the local field of forward accelerations associated with the diffusion process, which we constrain by demanding (see e.g. Refs. [35, 32, 33, 34] for prototypes of such dynamical constraints):

$$(D_+^2)\vec{X})(t) = (D_+\vec{b})(\vec{X}(t),t) = (\partial_t \vec{b} + (\vec{b}\nabla)\vec{b} + \nu \triangle \vec{b})(\vec{X}(t),t) = \vec{F}(\vec{X}(t),t) \tag{8}$$

where, at the moment arbitrary, function $\vec{F}(\vec{x},t)$ may be interpreted as the external forcing applied to the diffusing system, [29]. In particular, if we assume that drifts remain gradient fields, $curl\, \vec{b} = 0$, under the forcing, then those that are allowed by the prescribed choice of $\vec{F}(\vec{x},t)$ *must* fulfill the compatibility condition (notice the conspicuous absence of the standard Newtonian minus sign in this analogue of the second Newton law)

$$\vec{F}(\vec{x},t) = \nabla \Omega(\vec{x},t) \tag{9}$$

$$\Omega(\vec{x},t) = 2\nu[\partial_t \Phi \; + \; \frac{1}{2}(\frac{b^2}{2\nu} + \nabla b)]$$

establishes the Girsanov-type martingale connection of the forward drift $\vec{b}(\vec{x},t) = 2\nu\nabla\Phi(\vec{x},t)$ with the (Feynman-Kac, cf. [29, 28]) potential $\Omega(\vec{x},t)$ of the chosen external force field.

One of distinctive features of Markovian diffusion processes with the positive density $\rho(\vec{x},t)$ is that the notion of the *backward* transition probability density $p_*(\vec{y},s,\vec{x},t)$ can be consistently introduced on each finite time interval, say $0 \leq s < t \leq T$:

$$\rho(\vec{x},t)p_*(\vec{y},s,\vec{x},t) = p(\vec{y},s,\vec{x},t)\rho(\vec{y},s) \tag{10}$$

so that $\int \rho(\vec{y},s)p(\vec{y},s,\vec{x},t)d^3y = \rho(\vec{x},t)$ and $\rho(\vec{y},s) = \int p_*(\vec{y},s,\vec{x},t)\rho(\vec{x},t)d^3x$. This allows to define the backward derivative of the process in the conditional mean (cf. [35, 29, 36, 37] for a discussion of these concepts in case of the most traditional Brownian motion and Smoluchowski-type diffusion processes)

$$lim_{\triangle t\downarrow 0}\frac{1}{\triangle t}[\vec{x} - \int p_*(\vec{y},t-\triangle t,\vec{x},t)\vec{y}d^3y] = (D_-\vec{X})(t) = \vec{b}_*(\vec{X}(t),t) \tag{11}$$

$$(D_-f)(\vec{X}(t),t) = (\partial_t + (\vec{b}_*\nabla) - \nu\triangle)f(\vec{X}(t),t)$$

Accordingly, the backward version of the dynamical constraint imposed on the acceleration field reads

$$(D_-^2\vec{X})(t) = (D_+^2\vec{X})(t) = \vec{F}(\vec{X}(t),t) \tag{12}$$

where under the gradient-drift field assumption, $curl\,\vec{b}_* = 0$ we have explicitly fulfilled the forced Burgers equation (cf. (1)):

$$\partial_t\vec{b}_* + (\vec{b}_*\nabla)\vec{b}_* - \nu\triangle\vec{b}_* = \vec{F} \tag{13}$$

where, [32, 33, 29], in view of $\vec{b}_* = \vec{b} - 2\nu\nabla ln\rho$, we deal with $\vec{F}(\vec{x},t)$ previously introduced in (9). A notable consequence of the involved backward Itô calculus is that the Fokker-Planck equation (6) can be transformed to an *equivalent* form of:

$$\partial_t\rho(\vec{x},t) = -\nu\triangle\rho(\vec{x},t) - \nabla[\vec{b}_*(\vec{x},t)\rho(\vec{x},t)] \tag{14}$$

with the very same initial (Cauchy) data $\rho_0(\vec{x}) = \rho(\vec{x},0)$ as in (6).

At this point let us recall that equations (5) and (6) form a natural adjoint pair of equations that determine the Markovian diffusion process in the chosen time interval $[0,T]$. Clearly, an adjoint of (14), reads:

$$\partial_s f(\vec{x},s) = \nu\triangle f(\vec{x},s) - [\vec{b}_*(\vec{x},s)\nabla]f(\vec{x},s) \tag{15}$$

where:

$$f(\vec{x}, s) = \int p_*(\vec{y}, 0, \vec{x}, s) f(\vec{y}, 0) d^3y \ , \tag{16}$$

to be compared with (4),(5) and the previously mentioned passive scalar dynamics (2), see e.g. also [21]. Here, manifestly, the time evolution of the backward drift is governed by the Burgers equation, and the diffusion equation (15) is correlated (via the definition (10)) with the probability density evolution rule (14).

This pair *only* can be consistently utilized if the diffusion proces is to be driven by forced (or unforced) Burgers velocity fields.

Let us point out that the study of diffusion in the Burgers flow may begin from first solving the Burgers equation (12) for a chosen external force field, next specifying the probability density evolution (14), eventually ending with the corresponding "passive contaminant" concentration dynamics (15), (16). All that remains in perfect agreement with the heuristic discussion of the concentration dynamics given in Ref. [16], Chap. 7.3. where the "backward dispersion" problem with "time running backwards" was found necessary to *predict* the concentration.

Let us notice that the familiar logarithmic Hopf-Cole transformation, [2, 39], of the Burgers equation into the generalised diffusion equation (yielding explicit solutions in the unforced case) has received a generalisation in the framework of the so called Schrödinger boundary-data (interpolation) problem, [33, 34, 28, 29, 37, 38], see also [40, 41]. In particular, in its recent reformulation,[29, 28], the problem of defining a suitable Markovian diffusion process was reduced to investigating the adjoint pairs of parabolic partial differential equations, like e.g. (5), (6) or (14), (15). In case of gradient drift fields this amounts to checking (this imposes limitations on the admissible force field potential) whether the Feynman-Kac kernel

$$k(\vec{y}, s, \vec{x}, t) = \int exp[-\int_s^t c(\omega(\tau), \tau) d\tau] d\mu_{(x,t)}^{(y,s)}(\omega) \tag{17}$$

is positive and continuous in the open space-time area of interest, and whether it gives rise to positive solutions of the adjoint pair of generalised heat equations:

$$\partial_t u(\vec{x}, t) = \nu \triangle u(\vec{x}, t) - c(\vec{x}, t) u(\vec{x}, t) \tag{18}$$

$$\partial_t v(\vec{x}, t) = -\nu \triangle v(\vec{x}, t) + c(\vec{x}, t) v(\vec{x}, t)$$

where $c(\vec{x}, t) = \frac{1}{2\nu}\Omega(\vec{x}, t)$ follows from the previous formulas. In the above, $d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega)$ is the conditional Wiener measure over sample paths of the standard Brownian motion.

Solutions of (18), upon suitable normalisation give rise to the Markovian diffusion process with the factorised probability density $\rho(\vec{x}, t) = u(\vec{x}, t) v(\vec{x}, t)$ wich interpolates between the boundary density data $\rho(\vec{x}, 0)$ and $\rho(\vec{x}, T)$, with the forward and

backward drifts of the process defined as follows:

$$\vec{b}(\vec{x},t) = 2\nu \frac{\nabla v(\vec{x},t)}{v(\vec{x},t)} \tag{19}$$

$$\vec{b}_*(\vec{x},t) = -2\nu \frac{\nabla u(\vec{x},t)}{u(\vec{x},t)}$$

in the prescribed time interval $[0,T]$. The transition probability density of this process reads:

$$p(\vec{y},s,\vec{x},t) = k(\vec{y},s,\vec{x},t)\frac{v(\vec{x},t)}{v(\vec{y},s)} \ , \tag{20}$$

Here, neither k, (17), nor p, (20) need to be the fundamental solutions of appropriate parabolic equations, see e.g. ref. [28] where an issue of differentiability is analyzed.

The corresponding (since $\rho(\vec{x},t)$ is given) transition probability density, (10), of the backward process has the form:

$$p_*(\vec{y},s,\vec{x},t) = k(\vec{y},s,\vec{x},t)\frac{u(\vec{y},s)}{u(\vec{x},t)} \ . \tag{21}$$

Obviously, [28, 33], in the time interval $0 \leq s < t \leq T$ there holds:
$u(\vec{x},t) = \int u_0(\vec{y})k(\vec{y},s,\vec{x},t)d^3y$ and $v(\vec{y},s) = \int k(\vec{y},s,\vec{x},T)v_T(\vec{x})d^3x$.

By defining $\Phi_* = log\, u$, we immediately recover the traditional form of the Hopf-Cole transformation for Burgers velocity fields: $\vec{b}_* = -2\nu\nabla\Phi_*$. In the special case of the standard free Brownian motion, there holds $\vec{b}(\vec{x},t) = 0$ while $\vec{b}_*(\vec{x},t) = -2\nu\nabla log\, \rho(\vec{x},t)$.

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